COMPUTERS, MATHEMATICS EDUCATION, AND THE ALTERNATIVE EPISTEMOLOGY OF THE CALCULUS IN THE *YUKTIBHĀṢĀ*

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Introduction

The East-West Civilizational Clash in Mathematics: Pramāṇa *versus Proof*

In Samuel P. Huntington's terminology of a clash of civilizations, one might analyze the basis of the East-West civilizational clash as follows: the Platonic tradition is central to the West, even if we do not go to the extreme of Alfred North Whitehead's remark characterizing all Western philosophy as no more than a series of footnotes to Plato. But the same Platonic tradition is completely irrelevant to the East.

In the present context of mathematics, the key issue concerns Plato's dislike of the empirical, so the civilizational clash is captured by the following central question: *can a mathematical proof have an empirical component?*

The Platonic and Neoplatonic Rejection of the Empirical

According to university mathematics, as currently taught, the answer to the question above is no. Present-day university mathematics has been enormously influenced by (David Hilbert's analysis of) "Euclid's" *Elements.* Proclus,[1] a Neoplatonist and the first actual source of the *Elements,* argued that

> Mathematics ... occupies the middle ground between the partless realities ... and divisible things. The unchangeable, stable and incontrovertible character of [mathematical] propositions shows that it [mathematics] is superior to the kinds of things that move about in matter.... Plato assigned different types of knowing to ... the ... grades of reality. To indivisible realities he assigned intellect, which discerns what is intelligible with simplicity and immediacy, and ... is superior to all other forms of knowledge. To divisible things, in the lowest level of nature, that is, to all objects of sense-perception, he assigned opinion, which lays hold of truth obscurely, whereas to intermediates, such as the forms studied by mathematics, which fall short of indivisible but are superior to divisible nature, he assigned understanding.

In Plato's simile of the cave, the Neoplatonists placed the mathematical world midway between the empirical world of shadows and the real world of the objects that cast the shadows. Mathematical forms, then, were like the images of these objects in water—superior to the empirical world of shadows but inferior to the ideal world of the intellect, which could perceive the objects themselves.

Proclus explains that the term "mathematics" means, by derivation, the science of learning, and that learning (μα'θησις) is but a recollection of the knowledge that the soul has from its previous births, which it has forgotten—as Socrates had demonstrated with the slave boy. Hence, for Proclus, the object of mathematics is "to bring to light concepts that belong essentially to us" by taking away "the forgetfulness and ignorance that we have from birth" and reawakening the knowledge inherent in the soul. Hence Proclus values mathematics (especially geometry) as a spiritual exercise, like *hatha yoga,* which turns one's attention inward, away from sense perceptions and empirical concerns, and "moves our souls toward Nous" (the source of the light that illuminates the objects of which one normally sees only shadows and which one could better understand through their reflections in water).

In regarding mathematics as a spiritual exercise that helped the student to turn away from uncertain empirical concerns to eternal truths, Proclus was only following Plato. The young men of Plato's *Republic* (526 et seq.) were required to study geometry because Plato thought that the study of geometry uplifts the soul. Plato thought that geometry being a knowledge of what eternally exists, the study of geometry compels the soul to contemplate real existence; it tends to draw the soul toward truth. Plato emphatically added, "if it [geometry] only forces the changeful and perishing upon our notice, it does not concern us,"[2] leaving no ambiguity about the purpose of mathematics education in the Republic.

Rejection of the Empirical in Contemporary Mathematics

A more contemporary reason to reject any role for the empirical in mathematics is that the empirical world has been regarded as contingent in Western thought. Any proposition concerning the empirical has therefore been regarded as a proposition that can at best be *contingently* true. Hence, such propositions have been excluded from mathematics, which, it has been believed, deals only with propositions that are *necessarily* true: either eternally true, or at least true for all future time, or true in all possible worlds.[3]

In the twentieth century, it has, of course, again been (partly) accepted that mathematical theorems are not absolute truths,[4] but are true relative to the axioms of the underlying mathematical theory. Nevertheless, the relation between the axioms and theorems is still regarded as one of necessity: the theorems are believed to be *necessary* consequences of the axioms—it is believed that every possible (logical) world in which the axioms are true is a world in which the theorems are also true. A mathematical theorem such as $2 + 2 = 4$ is no longer regarded as eternally true, but, since this theorem can be *proved,* since it can be logically deduced from Peano's axioms, it is believed that $2 + 2 = 4$ is a necessary and certain *consequence* of Peano's axioms. It is today believed that although neither any axiom nor the theorem can be called a "necessary truth," the relation between axioms and the theorem can be so called. A theorem being the last sentence of a proof, theorems relate to axioms through the notion of mathematical proof, which is believed to embody and formalize the notion of logical necessity. Contemporary Western mathematics has not abandoned the notion of "necessary truth"; it has merely shifted the locus of this "necessary truth" from theorems and axioms to proof. From this perspective, admitting the empirical into mathematical proof would weaken and make contingent the relation of theorems to axioms, so that the empirical is still not allowed any place in the formal mathematical demonstration called "proof."

The current definition of a formal mathematical proof as enunciated by David Hilbert may be found in any elementary text on mathematical logic.[5] This definition may be stated informally as follows. A mathematical proof consists of a finite sequence of statements each of which is either an axiom or is derived from two preceding axioms by the use of modus ponens or some similar rules of reasoning. Modus ponens refers to the usual rule: $A, A \Rightarrow B$, hence B. The other "similar rules of reasoning" must be prespecified, and may include simple rules such as instantiation (for all x, $f(x)$, hence $f(a)$) and universalization ($f(x)$, hence for all x, $f(x)$), and so forth. A mathematical proof being such a sequence of statements, a reference to the empirical cannot be introduced in the course of a proof.

Neither can there be any reference to the empirical in the axioms at the beginning of a proof. Here, the word "axiom" is used in the sense of "postulate"; axioms are not regarded as self-evident truths but are merely an in-principle arbitrary set of propositions whose *necessary* consequences are explored in the mathematical theory. Since there is no reference here to the empirical, mathematical postulates and the primitive undefined symbols they involve are regarded, in principle, as being completely devoid of meaning.

Postulates relating to the empirical world lead to a physical theory and not to mathematics. This difference between mathematical and physical theories is embodied also in Karl Popper's criterion of refutability as follows. The theorems of the sentence calculus are exactly the tautologies. Although these tautologies may not be obvious, being tautologies they are not refutable. Unlike a mathematical theory, a physical theory must be (logically) refutable, and hence must contain some hypotheses and conclusions that are not tautologies. Mathematics concerns the tautologous relation between hypothesis and conclusions, while physics involves the empirical validity of the hypothesis/conclusions. Thus, no mathematical theory is a physical theory according to this widely used current philosophical classification, since no mathematical theory involves the empirical.

Acceptance of the Empirical in Indian Thought

However deeply rooted this rejection of the empirical may be, in Western ways of thinking about mathematics it seems to have gone unnoticed that not all cultures subscribe to this elevation of metaphysics above physics. Not all cultures and philosophies subscribe to this belief that the empirical world is contingent, and that only the nonempirical can be necessary. For example, the Lokāyata (popular/materialist) stream of thought in India adopts exactly the opposite point of view. It explicitly rejects any world except that of sense perception. It admits the *pratyakṣa* or the empirically manifest as the only sure means of *pramāṇa* or validation, while rejecting *anumāna* or inference as error-prone and fallible. That is, in terms of the Platonic gradation of reality, Lokāyata places intellectual ways of knowing on a *lower* footing than knowledge relating directly to sense perception. However odd this may seem from a Western perspective, and notwithstanding the orientalist characterization of Indian thought as "spiritual," all major Indian schools of thought concur in accepting the *pratyakṣa* as a valid *pramāṇa,* or means of validation. Moreover, *pratyakṣa* is the sole *pramāṇa* that is so accepted by all schools, since Lokāyata rejects *anumāna,* while Buddhists accept *anumāna* but reject *śabda* or authoritative testimony, although Naiyāyikas accept all three, and add the fourth category of analogy *(upamāna).*

The *pratyakṣa* enters explicitly also into the mathematical rationale, in the Indian way of doing mathematics from the time of the *śulba sūtras* (ca. 600 C.E.),[6] through Aryabhata (ca. 500 C.E.),[7] and up to the time of the *Yuktibhāṣā* (ca. 1530 C.E.).[8] For example, the geometry of the *śulba sūtras*, as the name suggests, involves a rope *(śulba)* for measurement. Aryabhata defines water level as a test of horizontality, and the plumb line as the test of perpendicularity (*Gaṇita* 13):

> The level of ground should be tested by means of water, and verticality by means of a plumb.

The *Yuktibhāṣā* proves the "Pythagorean" "theorem"[9] in one step by drawing a diagram on a palm leaf, cutting along a line, picking, and carrying. The rationale is

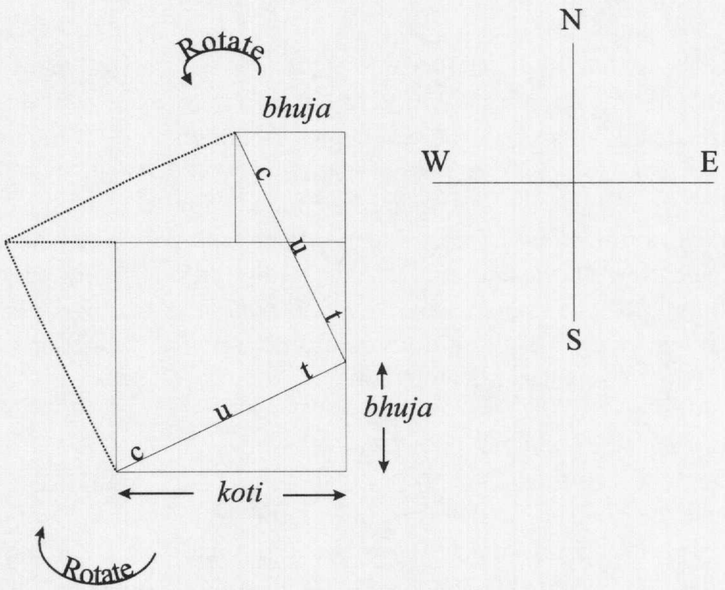

Fig. 1. Rationale for the sine rule in the *Yuktibhāṣā.* The square corresponding to the smaller side *(bhuja)* is drawn on a palm leaf and placed on the square corresponding to the bigger side *(koti),* as shown. The *bhuja* is measured off from the SE corner of the larger square, and joined to the SW corner of the larger square and the NW corner of the smaller square. Cutting along the joining lines and rotating gives the square on the hypotenuse. This simple proof of the "Pythagorean" "Theorem" involves (a) measurement and (b) movement of the figure in space.

explained in figure 1: the diagram is to be drawn on a palm leaf, and, as indicated, it is to be measured, cut, and rotated. The details of this rationale are not our immediate concern beyond observing that drawing a figure, carrying out measurements, cutting, and rotation are all empirical procedures. Hence, such a demonstration would today be rejected as invalid solely on the ground that it involves empirical procedures that *ought* not to be any part of mathematical proof.

Genesis of the Current Notion of Mathematical Proof: SAS and the Empirical

Paradoxically, although the currently dominant notion of mathematical proof, as formulated by Hilbert at the turn of the twentieth century, is essentially modeled on "Euclid's" *Elements,* the empirical is not entirely rejected in the *Elements.* Hence, a brief reexamination of the history of the *Elements* is in order, to illuminate the historical process by which the empirical was eventually eliminated from Western mathematics.

Proclus of Alexandria (fifth century C.E.), the earliest source to mention "Euclid" the geometer, not only believed in the "unchanging, stable and incontrovertible" character of mathematics; he started his commentary by pointing to the persuasiveness of the *Elements:*

> Euclid ... put together the *Elements* ... bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors.[10]

This element—the persuasiveness of the reasoning in the *Elements* and its apparent incontrovertibility—was picked up successively by Islamic and Christian rational theology, which last abandoned the Neoplatonic and Islamic concern with equity.[11] Thus, the *Elements* came to be valued in the West chiefly for the orderly arrangement of the theorems—each theorem depended only on what had previously been established, as in the present-day notion of mathematical proof—which brought the theorems of geometry to "irrefragable demonstration." The persuasiveness of the *Elements* was a key concern for Islamic rational theology, and it became the sole concern for Christian rational theology, since the unbeliever (or opponent), who did not accept the scripture (or its interpretation), nevertheless accepted reason. "Mathematically proved" is, even today, virtually synonymous with "incontrovertible." In Christian rational theology this was in contrast to empirical procedures that were *not* "incontrovertible," since the empirical world had to be regarded as contingent.[12]

But, while Proclus regarded mathematics as a means of moving away from the empirical, he did *not* regard mathematics as *disjoint* from the empirical; he did *not* think that the empirical had no role at all in a proof—he thought that a proof must suit the thing to be proved:

> Proofs must vary with the problems handled and be differentiated according to the kinds of being concerned, since mathematics is a texture of all these strands and adapts its discourse to the whole range of things.[13]

Since Proclus accorded to mathematics an intermediate status, between the gross empirical world and the higher Platonic world of ideals, *Proclus was ready to accept the empirical at the beginning of mathematics,* just as much as he was ready to accept that diagrams had an essential role in mathematical proof, to stir the soul from its forgetful slumber.[14] Proclus thought that the "sensible" (visible) line in a diagram served to remind the viewer of an "intelligible" line—hence, the "sensible" line could not possibly be substituted by a beer mug, which would remind the viewer of something else. Therefore, historically speaking, until the twentieth century the *Elements* had at least one *essential* reference to the empirical in Proposition I.4.

This reference to the empirical in *Elements* I.4 was subsequently eliminated following Hilbert,[15] Russell,[16] and others, who suggested that "Euclid" had made a mistake in proving the theorem. Hence, that *theorem* was incorporated as the SAS (Side-Angle-Side) *postulate*, today taught in school geometry.[17] The theorem asserts that if two sides and the included angle (Side-Angle-Side) of one triangle are equal to those of another triangle, then the two triangles are equal ("congruent" in Hilbert's terminology—which bypassed also the political significance of equity in the *Elements,* which was a key aspect of the *Elements* for Neoplatonists and Islamic rational theologians). The proof of this theorem, as actually found in the manuscripts of the *Elements,* involves picking one triangle, moving it, and placing it on top of the other triangle to demonstrate the equality—an empirical procedure similar to that used in the *Yuktibhāṣā* proof of the "Pythagorean" "Theorem." The proofs of subsequent theorems of the *Elements,* however, avoid this empirical process, with the possible exception of I.8.

The question before us is this: is it legitimate to accept the empirical at one point in mathematical discourse and to reject it elsewhere?

From the point of view of Proclus, the appeal to the empirical in the proof of I.4 was acceptable, since proofs must be differentiated according to the kinds of being, and the empirical was the starting point of mathematics, although not its goal. Empirical procedures were therefore acceptable in proofs at the beginning of mathematics, although the proofs of subsequent propositions must move away from the empirical, to suit the objectives of mathematics. For Hilbert, who sought the standardization and consistency suited to an industrial civilization, a notion of mathematical proof that varied according to theorems, or "kinds of beings," was not acceptable. Indeed, in Hilbert's time, in the West, industrialization was practically synonymous with civilization, as in the statement "Civilization disappears ten feet on either side of the railway track in India." So it is no surprise that Hilbert's view of mathematics was entirely mechanical[18]—where Proclus sought to persuade human beings, Hilbert sought to persuade machines! Hilbert's notion of proof, therefore, had to be acceptable to a machine; a proof had to be so rigidly rule-bound that it could be mechanically checked—an acceptable proof had to be acceptable in *all* cases. Hence, exceptions do not prove the rule; a single exception disproves the rule—a belief that is the basis also of Popper's criterion of falsifiability. Hence, Hilbert et al. chose to reject as unsound the proof of *Elements* I.4.

In rejecting the traditional demonstration of *Elements* I.4, Hilbert also reflected the then-prevalent Western view that doubted the role of measurement and the empirical in mathematics. Picking and carrying involves movement in space, and it was thought that movement in space may deform the object, much as a shadow moving over uneven ground may be deformed. The avoidance of picking and carrying in the proofs of the subsequent theorems in the *Elements* was interpreted, by the twentieth century, as an implicit expression of doubt about the very possibility of measurement. It was argued against Helmholtz that measurement required (a) the notion of motion, and furthermore that this motion must be without deformation, so that it required (b) the notion of a rigid body—and neither of these was the proper concern of geometry, which ought to be concerned only with motionless space. (The notion of a rigid body depends on physical theory; e.g., the Newtonian notion of a rigid body has no place in relativity theory, for such a rigid body would allow signals to propagate at infinite speed.)

Since measurement, for example of length, involves moving one object to bring it in coincidence with another, the doubt about measurement was expressed as a doubt about (a) the role of motion in the foundations of mathematics, and (b) the possibility and meaning of motion without deformation. In favor of (a) the authority of Aristotle was invoked to argue that motion concerned astronomy and that mathematics was "in thought separable from motion." The authority of Kant was implicitly invoked to argue that motion was not a priori, but involved the empirical, and *hence* could not be part of mathematics. All these worries are captured in Schopenhauer's criticism of the "Theonine" Axiom 8 (corresponding to the "Heiberg" Common Notion 4), which supports SAS:

> ... *coincidence* is either mere tautology, or something entirely empirical, which belongs not to pure intuition, but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus, this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and empirical.[19]

In short, motion, with or without deformation, brought in empirical questions of physics, and Plato, Aristotle, and Kant all concurred that mathematics *ought* not to depend upon physics, but *ought* to be a priori, and that geometry *ought* to be concerned only with immovable space. Hence the proof of SAS (*Elements* I.4) came to be regarded as unacceptable, and the status of SAS was changed from a theorem to a postulate.

The Epicurean Ass

As already observed above, the requirement of a consistent notion of proof limited Hilbert's options. If an appeal to the empirical is permissible in the proof of *one* theorem (*Elements* I.4), then why not permit an appeal to the empirical in the proof of *all* theorems? Why not permit triangles to be moved around in space to prove the "Pythagorean" theorem (*Elements* 1.47), as in the *Yuktibhāṣā* proof? Why not permit length measurements? Accepting the empirical as a means of proof (or even introducing a measure of length axiomatically, as done by Birkhoff[20]) greatly simplifies the proofs of the theorems in the *Elements.* In fact, so greatly does it simplify the proofs that it makes most of the theorems of the *Elements* obvious and trivial! Since the indigenous Indian tradition of geometry relied on measurement, one strand of

Indian tradition rejected the *Elements* as valueless from a practical standpoint until the mid-eighteenth century, when they were first translated from Persian into Sanskrit by Jaisingh.

That the *Elements* are trivialized by the consistent acceptance of the empirical definitely was the basis of the objections raised by the Epicureans, who may be regarded as the counterpart of the Lokāyata in the Greek tradition. The Epicureans argued, against the followers of "Euclid," that the theorems of "Euclid's" *Elements* were obvious even to an ass. They particularly referred to *Elements* I.20, which asserts that in any triangle the two sides taken together in any manner are greater than the third. The Epicureans argued that any ass knew the theorem since the ass went straight to the hay and did not follow a circuitous route, along two sides of a triangle. Proclus replied that the ass only knew *that* the theorem was true; it did not know *why* it was true.

The Epicurean response to Proclus has, unfortunately, not been well documented. The Epicureans presumably objected that mathematics could *not* hope to explain *why* the theorem was true, since mathematics was ignorant of its own principles. They presumably quoted Plato (*Republic* 533):

> [G]eometry and its accompanying sciences ... —we find that though they may dream about real existence, they cannot behold it in a waking state, so long as they use hypotheses which they leave unexamined, and of which they can give no account. For when a person assumes a first principle which he does not know, on which first principle depends the web of intermediate propositions and the final conclusion—by what possibility can such mere admission ever constitute science?[21]

It is to this objection that Proclus presumably responds when he asserts that Plato does not declare that

> mathematics [is] ignorant of its own principles, but says rather that it takes its principles from the highest sciences and, holding them without demonstration, demonstrates their consequences.[22]

This appeal to Plato's authority, and to the Platonic gradation of the sciences, is obviously inadequate to settle the issue—for the Lokāyata would reject as non-science what Plato regards as the "highest science" (although they would have agreed with Proclus about equity). Contrary to Plato, the Lokāyata would insist that mathematics must take its principles from the empirical world of sense perceptions, a move that would also destroy the difference between mathematics and physics in current Western philosophical classification.

Although Proclus has gone largely unanswered down through the centuries—presumably because no Epicureans were left to respond to him—the present essay will provide an answer from the perspective of traditional Indian mathematics.

Mathematics-as-Calculation versus Mathematics-as-Proof

The trivialization of the *Elements* by the acceptance of the empirical can be viewed from another angle: what is mathematics good for? Why *do* mathematics? As already stated, Proclus explains at great length in his introduction to the *Elements* that although (a) mathematics has numerous practical applications, (b) mathematics must be regarded primarily as a spiritual exercise. Thus, Proclus states:

> Geodesy and calculation are analogous to these sciences [geometry, arithmetic], . . . [but] they discourse not about intelligible but about sensible numbers and figures. For it is not the function of geodesy to measure cylinders or cones, but heaps of earth considered as cones and wells considered as cylinders; and it does not use intelligible straight lines, but sensible ones, sometimes more precise ones, such as rays of sunlight, sometimes coarser ones, such as a rope or a carpenter's rule.[23]

Clearly, for Proclus, the practical applications of mathematics were its lowest applications, involving "sensible" objects rather than "intelligible" objects:

> [I]nstead of crying down mathematics for the reason that it contributes nothing to human needs—for in its lowest applications, where it works in company with material things, it does aim at serving such needs—we should, on the contrary, esteem it highly because it is above material needs and has its good in itself alone.[24]

This echoes the Platonic deprecation of the applications of mathematics (*Republic* 527):

> They talk, I believe in a very ridiculous and poverty-stricken style, for they speak invariably of squaring and producing and adding, and so on, as if they were engaged in some business, and as if all their propositions had a practical end in view: whereas in reality I conceive that the science is pursued wholly for the sake of knowledge.[25]

Plato clearly thought of mathematics-as-calculation as distinctly below mathematics-as-proof, and this Platonic valuation led to the implicit valuation of pure mathematics as superior to applied mathematics, and to the resulting academic vanity of pure mathematicians, who regarded (and still regard) themselves as superior to applied mathematicians—a vanity so amusingly satirized in Swift's *Gulliver's Travels.*

In traditional Indian mathematics, however, there never was such a conflict between "pure" and "applied" mathematics, since the study of mathematics was never an end in itself, but was always directed to some other practical end. Geometry, in the *śulba sūtra* was not directed to any spiritual end, but to the practical end of constructing a brick structure. Contrary to Plato, calculation was valued and taught as much for its use in commercial transactions as for its use in astronomy and timekeeping. Proof was not absent, but it took the form of a rationale for methods of calculation. The methods of calculation were regarded as valuable, not the proofs by themselves—there was no pretense that rationale provided any kind of absolute certainty or necessary truth. Rationale was not valued for its own sake. Hence, rationale was not considered worth recording in many of the terse (*sūtra*-style) authoritative texts on mathematics, astronomy, and timekeeping. On the other hand, rationale was not absent, but was taught, as is clear, for example, from the very title *Yuktibhāṣā,* or, in full form, the *GaṇitaYuktiBhāṣā,* which means "discourse on rationale in mathematics."

The Epistemological Discontinuity

These differing perceptions of the nature and purpose of mathematics had interesting consequences when the two streams of mathematics collided. It is natural that those who valued the practical applications of mathematics—the Florentine merchants—played a major role in importing the Indian techniques of calculation into Europe, as algorismus texts. (Algorismus, as is well known, is a Latinization of al Khwarizmi.) The mathematical epistemology underlying the algorismus texts—Latin translations of al Khwarizmi's Arabic translation of Brahmagupta's Sanskrit manuscript—contrasted sharply with the medieval European view of mathematics, and the contrasting epistemologies led to major difficulties, such as the difficulty in understanding *śūnya*—non-representable—ultimately interpreted as zero. Although the practical applications of mathematics were valued de facto in the West, so enormous were the difficulties that the West had in understanding the Indian tradition of mathematics that the acceptance of algorismus texts in Europe took around *five centuries,*[26] from the first recorded algorismus text by the tenth-century Gerbert (Pope Sylvester II) to the eventual triumph of algorismus techniques as depicted on the cover of Gregor Reisch's *Margarita Philosophica.*[27] Indeed, the British Treasury continued to use the competing abacus techniques as late as the eighteenth century, until, in the nineteenth century, the practice was forcibly ended by burning all tally sticks—and in the process also burning down the British parliament!

It is less well known that a similar epistemological discontinuity arose in connection with the calculus. The "Pythagorean" theorem is merely the starting point of the *Yuktibhāṣā,* which goes on to develop infinite series expansions for the sine, cosine, and arctan functions, nowadays known as the "Taylor" series expansions, to calculate very precise numerical values for the sine and cosine functions. In the sixteenth century, Indian mathematical and astronomical manuscripts engaged the attention of Jesuit priests,[28] because of their practical application to navigation through astronomy and timekeeping. Christoph Clavius, who reformed the Jesuit mathematical syllabus at the Collegio Romano, emphasized the practical applications of mathematics. A student and later correspondent of the famous navigational theorist Pedro Nunes, Clavius understood the relation of the date of Easter to latitude determination through the measurement of the solar altitude at noon, as described in the texts of Bhaskara I—the *Mahābhāskarīya* and the very widely distributed *Laghu Bhāskarīya.*[29] In his role as head of the committee for the reform of the Gregorian calendar, Clavius presumably received inputs from students like Matteo Ricci whom he had trained in mathematics, astronomy, and navigation. Ricci later went to Cochin, and wrote back that he was seeking to learn the methods of timekeeping from "an intelligent Brahman or an honest Moor."[30] (The Jesuits, of course, knew Malayalam, the language of the *Yuktibhāṣā;* they had even started printing presses in Malayalam by then, and were teaching Malayalam to the locals in the Cochin college, at the latest by 1590.)

The calculus was the key technique needed to determine precise sine values as in the *Yuktibhāṣā.* Precise sine values were needed for various purposes in navigation—to calculate loxodromes, for example—hence, precise sine values were a key

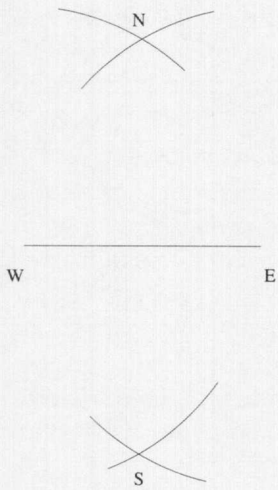

Fig. 2. The fish figure and *Elements* I.1. With W as center and WE as radius, two arcs are drawn, and they intersect the arcs drawn with E as center and EW as radius at N and S. The construction above, called the fish figure, is used in India to construct a perpendicular bisector to the EW line and thus determine NS. In the *Elements* I.1, a similar construction is used to construct the equilateral triangle WNE on the given segment WE. Although it is empirically manifest *(pratyakṣa)* that the two arcs must intersect in a point, to *prove* their intersection, without appeal to the empirical, real numbers are required, for, with rational numbers, the two arcs may "pass through" each other, without there being any point at which they intersect, since there are "gaps" in the arcs, corresponding to the "gaps" in rational numbers.

concern of European navigational theorists and astronomers like Nunes, Mercator, Simon Stevin,[31] and Christoph Clavius,[32] who provided their own sine tables.

The computation of precise sine values is closely related to the numerical determination of the length of the arc. The contrasting epistemologies of Indian and Western mathematics, however, led to another protracted epistemological struggle. For example, Descartes declared in his *La Geometrie* that determining the ratio of a curved to a straight line was intrinsically impossible. His contemporaries, and other participants in Mersenne's discussion group, like Pascal and Fermat, believed to the contrary, and used the "infinite series," almost exactly as in the *Yuktibhāṣā,* to calculate the length of the arc of "parabolas" of all orders. This procedure involved "infinitesimals" and "infinities" and initiated the protracted epistemological struggle in Europe concerning the meaning and nature of infinitesimals. It was only toward the end of the nineteenth century that Dedekind's formulation of the real numbers partly resolved the issues regarding infinitesimals, while also clearing up the implicit and less-noticed reference to the empirical in the proof (fig. 2) of the very first proposition in the *Elements.*[33] This issue of infinities and infinitesimals, by the way, is not quite settled yet. Why not use non-Archimedean field extensions of the reals, not as in Non-standard analysis as an intermediate step, but really accepting infinitesimals and infinities as the case? Nor is the issue settled from the practical standpoint: the δ-function could only be partly formalized in the Schwartz theory of distributions, for it does not permit the multiplication of distributions, as in δ^2, which is crucial to the problem of infinities (the renormalization problem) in quantum field theory. I will not examine these more technical questions here—the point is simply that the story of the epistemological difficulties with the calculus has not quite reached a conclusion.

Toward an Alternative Epistemology of Mathematics

The present-day schism between mathematics-as-calculation and mathematics-as-proof is one of the consequences of the historical discontinuities and continuities discussed above: on the one hand the practical and empirical is rejected; on the other hand there is the persistent attempt to assimilate practical/empirical mathematics-as-calculation into spiritual/formal mathematics-as-proof. Practical mathematics, as in the Indian tradition, regarded mathematics as calculation, whereas the idea of mathematics as a spiritual exercise has developed into the current Hilbert-Bourbaki approach to mathematics as formal proof, which has dominated mathematical activity for most of the twentieth century. The attempt to assimilate practical and empirical mathematics into the tradition of spiritual and formal mathematics has gone on now for over a thousand years. However, despite the apparent epistemological satisfaction provided by mathematical analysis, for example, the calculus still remains the key tool for practical mathematical calculations, and few physicists or engineers, even today, study either Dedekind's formulation of real numbers or the more modern notion of integral and derivative—either the Lebesgue integral or the Schwartz derivative. The practical seems to get along perfectly well without the need for any metaphysical seals of approval!

This schism within mathematics is today being rapidly widened by the key technology of the twentieth century, the computer. The availability of this superb tool for calculation has accentuated the imbalance between mathematics-as-calculation and mathematics-as-proof. With a computer, numerical solutions of various mathematical problems can readily be calculated even though one may be quite unable to *prove* that a solution of the given mathematical problem exists or is unique. For example, today one can calculate on a computer the solution of a stochastic differential equation driven by Lévy motion, although one cannot today prove the existence and uniqueness of the solution. The advocates of mathematics-as-calculation suggest that the practical usefulness of the numerical solution—the ability to become rich through improved predictions of price variations in the stock market—overrides the loss of certainty in the absence of proof. The advocates of mathematics-as-proof argue that what lacks certainty cannot be mathematics, irrespective of its usefulness.

Is this schism in mathematics a "natural law"? Must useful mathematics remain epistemologically insecure for long periods of time? Or is this state of affairs the outcome of the actual history of mathematics? Can we understand the civilizational tensions that have determined the actual historical trajectory of mathematics, and modify mathematics to resolve these tensions? Can an alternative epistemology of mathematics be found that is better suited to mathematics-as-calculation? How should mathematics be taught under these circumstances of a widening gap between mathematics-as-calculation and mathematics-as-proof? I believe that the way to answer these questions is to probe the alleged epistemological security of mathematics-as-proof by re-examining the very notion of mathematical proof—is mathematical "proof" synonymous with certainty?

Summary

To recapitulate, in mathematics, the East-West civilizational clash may be represented by the question of *pramāṇa* versus proof: is *pramāṇa* (validation), which involves *pratyakṣa* (the empirically manifest), not valid proof? The *pratyakṣa* is the one *pramāṇa* that is accepted by all major Indian schools of thought, and this is incorporated into the Indian way of doing mathematics, while the same *pratyakṣa,* since it concerns the empirical, is regarded as contingent and is entirely rejected in Western mathematics. Does mathematics relate to calculation, or is it primarily concerned with proving theorems? Does the Western idea of mathematical proof capture the notions of "certainty" or "necessity" in some sense? Should mathematics-as-calculation be taught primarily for its practical value, or should mathematics-as-proof be taught as a spiritual exercise?

Formal Mathematics as a Social Construction

In attempting to resolve this civilizational clash, the key question to examine is this: are mathematical theorems "necessary," are they universal truths, or are they merely social constructions? I will argue that the theorems of formal mathematics are social constructs, and that the belief in their validity or necessity rests on nothing more solid than social authority. Various arguments have been given in this direction, but I regard the arguments in the section below on "The Cultural Dependence of Logic" as conclusive.

Integers (Ints) and Real Numbers (Floats) on a Computer

For the purpose of clarifying the nature of formal mathematical theorems, let us take an apparently certain universal mathematical truth: $2 + 2 = 4$. Is $2 + 2 = 4$ a universal truth, or is it a social construction and hence a cultural truth? Perhaps one should first take up the easier case of $1 + 1$! The usual belief is that $1 + 1 = 2$. One could also amplify this belief negatively as what $1 + 1$ is not: if $1 + 1 = 2$ is a universal truth, then $1 + 1 = 0$ or $1 + 1 = 1$ or $1 + 1 = 3$ must all be universally false. However, if 0 and 1 denote truth values, we know, for instance, that $1 + 1 = 1$ holds in classical two-valued logic, with + denoting "*in*clusive or," 0 denoting "false," and 1 denoting "true." We know that $1 + 1 = 0$ holds in classical two-valued logic with + denoting "*ex*clusive or." And $1 + 1 = 0$ is also the case if 0 and 1 denote binary digits (bits) and + denotes addition with carry. This case is one that is commonly implemented thousands of times in the chips of a computer.

We see that if, at all, $1 + 1 = 2$ is a universal truth, it is at best a qualified universal truth. It is necessary to specify what 1, +, and = are; these are merely symbols that, lacking any empirical reference, could be performing multiple duties. Today we would tend to qualify that in $1 + 1 = 2$, the 1, +, =, and 2 relate to "natural numbers" or to integers or to rational numbers or real numbers. However, in current formal mathematics, since the axioms, lacking any empirical reference, are practically arbitrary, there can be no real restriction on how one specifies the syntactic rules for using 1, +, and =. To return to the harder case of $2 + 2$, it is perfectly possible, for example, in current formal mathematics, to specify 2, +, and = so that $2 + 2 = 5$. Thus, let $a + b = a @ b @ 1$, where @ is an unusual notation for usual addition (socially conventional addition in "natural numbers"). One cannot say that such a formal theory is useless, for like all pure mathematics it may find a use some day. (Indeed, it has a use already in philosophy for purposes of illustration!) At best, one can say that this or that mathematician who enjoys a certain degree of social recognition finds it uninteresting. So the theory of numbers with $2+2=5$ is not false; it is, at worst, a way to handle numbers that some existing social authorities may find socially uninteresting.

What is socially interesting or uninteresting can naturally vary with the cultural circumstances: for instance, $2+2=5$ may be a socially interesting case for native South Americans.[34]

What is socially interesting or uninteresting can also vary across time as technology varies. Computers are widely used today, but one cannot make a computer "understand" or work with natural numbers or real numbers. For the purposes of programming a computer, the standard convention is that an integer (int data type) is something that can be represented using two bytes or sixteen bits. Setting aside one bit to represent the sign (positive or negative), the largest (signed) integer that can then be represented is 111111111111111 (fifteen ones), in binary notation, or $2^{14}+2^{13}+\cdots+2^2+2^1+2^0=2^{15}-1=32767$. This convention suits the eight-bit architecture; but nothing will change, except the value of the upper limit, if we move from an eight-bit to a 128-bit machine, or use static storage, with any finite number of bits. The number 32767 may change as technology and conventions change, but the point is that for any computer whatsoever there will always be such an upper limit, so long as we are dealing with actual computers rather than abstract Turing machines with infinite memory, which are as imaginary and nonexistent as a barren woman's son or a rabbit with horns.

The existence of an upper limit creates a serious problem in computer arithmetic, relating to the Western mathematical conceptualization of "natural numbers," asserted by Dedekind to have been given by God. One can have $2+2=4$ on a computer, but only at the expense of admitting that

$$20{,}000+20{,}000=-25{,}536.$$

Anyone who disbelieves this is welcome to use the accompanying computer program (fig. 3) in the C language to check this out. One can represent the natural numbers needed for all or for most *practical* purposes, but one cannot represent the idea of a "natural number" on a computer, and one cannot represent addition according to Peano's axioms on a computer. *It is impossible to program the syntax of natural numbers on any actual computer.*

A desktop calculator usually manages to get the sum given above correctly. How is this achieved? One can get the expected answer by using floating-point numbers, which roughly correspond to real numbers. The upper limit becomes much higher, but we can now validly have

$$2+2=4.00000000000000001$$

with sixteen zeros, which is typically the case in a computer (which observes the IEEE standard[35] for floating-point arithmetic). From a practical standpoint, this arith-

```
/*Program name: addint.c
Function: To demonstrate how a computer adds integers */

#include <stdio.h>
#include <conio.h>

void main (void)
{
        int a, b, c;
        printf ("\n Enter a = ");
        scanf ("%d", &a);
        printf ("\n Enter b = ");
        scanf ("%d", &b);
        c = a+b;
        printf ("\n %d + %d = %d", a, b, c);
        getch();
        return;
}
```

Program Input and Output:

```
Enter a = 20000
Enter b = 20000
20000 + 20000 = -25536
```

Fig. 3. A C program to add two integers. The C program above shows how a computer adds two integers. If the program is compiled and run, the program output will be as shown. It is possible to do arithmetic to larger precision, but it is impossible to do the arithmetic of Peano's natural numbers on a computer.

metic is quite satisfactory. From the standpoint of the current formal mathematics of real numbers, this type of arithmetic only *seems* more satisfactory: serious problems arise because the equation above means that floating-point numbers do *not* obey the same algebraic rules as real numbers. The associative law, for example, fails for arithmetic operations with floating-point numbers. Thus,

$$(0.00000001 + 1) - 1 = 0$$

but

$$0.00000001 + (1 - 1) = 0.00000001$$

Once again, one can achieve a higher precision, one can arrange things so that in the equation above the number of zeros dazzles the eye. One can arrange for a number of decimal places adequate for all practical, physical, and engineering purposes. But one cannot bypass, in principle, the failure of the associative law. There will always remain not one or two but an uncountable infinity of "exceptions" to the associative law for addition. Similarly, the associative law and cancellation law for multiplication fail, and so does the distributive law linking addition and multiplication. Hence, the numbers on a computer can never correspond to the numbers in the formal systems of natural numbers or real numbers. Since computers are socially interesting, so are numbers not corresponding to natural or real numbers.

The other point I am trying to drive at is the following: real numbers may help to bypass the appeal to the real world in *Elements* I.1, but in the real (empirical) world, as distinct from some imagined or ideal Platonic world, there is no satisfactory way to *represent* the natural or real numbers, since there is no way to represent any real number with only a finite number of symbols. Hence also, there is no satisfactory way to *represent* the alleged universal truth that $2 + 2 = 4$, since there is no satisfactory way to *state* the required qualification that this equation concerns natural or real numbers. The representation of natural numbers according to Peano's axioms involves a supertask, an infinite series of tasks, usually hidden by the ellipsis but made evident by computer arithmetic, which can hence never be the arithmetic of Peano's natural numbers or Dedekind's real numbers.

For practical purposes, no supertask is necessary: the representation of numbers on a computer is satisfactory for mathematics-as-calculation, but it is unsatisfactory or "approximate" or "erroneous" from the standpoint of mathematics-as-proof. Indian mathematics, which dealt with "real numbers" from the very beginning ($\sqrt{2}$ finds a place in the *śulba sūtras*), does not represent numbers by assuming that such supertasks can be performed, any more than it represents a line as lacking any breadth, for the goals of mathematics in the Indian tradition were practical not spiritual. The Indian tradition of mathematics worked with a finite set of numbers, similar to the numbers available on a computer, and similarly adequate for practical purposes. Excessively large numbers, like an excessively large number of decimal places after the decimal point, were of little practical interest. Exactly what constitutes "excessively large" is naturally to be decided by the practical problem at hand, so that no universal or uniform rule is appropriate for it.

On the other hand, theoretically speaking, formal Western mathematics is not formulated with a view to solving practical problems: it treats both natural and real numbers from an idealist standpoint; hence it runs into the difficulty with supertasks, made evident by computer arithmetic.

To take stock, Plato and Proclus rejected the practical and empirical as valueless relative to the ideal; subsequent developments stripped away the spiritual and political content of Neoplatonic mathematics; formal mathematics has also discarded meaning and truth. The result is a formulation of elementary arithmetic that involves a supertask that no supercomputer will ever be able to perform. If mathematics exclusively concerns the impractical, the imaginary, the meaningless, and the arbitrary, then of what value is mathematics? Why should one continue to accept Plato's injunction to teach this sort of mathematics to one's children? The only potentially valuable element left in Western mathematics today is the notion of "proof." The notion of "proof" is the fulcrum of Western mathematics—the whole edifice of twentieth-century mathematics has been made to rest on the notion of mathematical proof.

The Cultural Dependence of Logic

One can inquire into the nature of this "proof" or criterion of validity. One can inquire into the cherished belief that mathematical proof involves only reason or logical deduction, which is universal and certain—for it is this belief that makes the notion of mathematical proof potentially valuable. Can one even maintain universality for the criteria of validity? Can one assert that there is a necessary relation between the meaningless and unreal assertion $2 + 2 = 4$ and the arbitrary set of axioms known as Peano's axioms? The short answer is no. The validation of $2 + 2 = 4$ requires proof—one is able to prove $2 + 2 = 4$ from Peano's axioms. But this proof relies on modus ponens; and modus ponens implicitly involves a notion of implication that requires two-valued logic.[36] Thus, the entire value of formal Western mathematics rests on the belief in the universality of a two-valued logic.

However, a two-valued logic, as I have repeatedly stressed,[37] is not universal. The belief in a truth-functional two-valued logic was denied by the Buddhists and Jains, for example.[38] Walshe[39] refers to this as "the four 'alternatives' of Indian logic: a thing (a) is, (b) is not, (c) both is and is not, and (d) neither is nor is not." This logic of four alternatives certainly did not apply to all Indian logic, but it was frequently used by Nāgārjuna in his famous tetralemma *(catuskoti).* This logic is illustrated by the following example from the *Brahmajāla Sutta* of the *Dīgha Nikāya.* This Sutta records the Buddha's discourse against various wrong views. The Buddha described four wrong views concerning the nature of the world—whether it is finite or infinite—whose adherents claim as follows:

> "I know that the world is finite and bounded by a circle." This is the first case. . . . "I know that this world is infinite and unbounded." This is the second case. And what is the third way? . . . "I . . . perceiv[e] the world as finite up-and-down, and infinite across. Therefore I know that the world is both finite and infinite." This is the third case. And what is the fourth case? Here a certain ascetic or Brahmin is a logician, a reasoner. Hammering it out by reason, he argues: "This world is neither finite nor infinite. Those who say it is finite are wrong, and so are those who say it is infinite, and those who say it is finite and infinite. This world is neither finite nor infinite." This is the fourth case. These are the four ways in which these ascetics and Brahmins are Finitists and Infinitists. . . . There is no other way.[40]

The four wrong views about the world, described by the Buddha are: (1) The world is finite; (2) the world is not finite; (3) the world is both finite and infinite; and (4) the world is neither finite nor infinite. The semantic interpretation of (3) is that the world is finite up-and-down and infinite across. The semantic interpretation of (4) is that all three of the preceding views are wrong; it is said to be "hammered out by reason." A fifth possibility was explicitly denied, although such a belief, too, was in vogue. Later on in the same *Brahmajāla Sutta,* the Buddha, like Ajatasattu, again rejects the use of more than four possibilities, describing them by the epithet: the "Wriggling of the Eel."[41]

Not too much should be read into the particular semantic interpretation for the case (3) above. Thus, Nāgārjuna, in his famous tetralemma *(catuskoti)* puts forward the proposition:

Everything is
such
not such
both such and not such
neither such nor not such.[42]

The writings of Dinnaga[43] on this point are a bit obscure, particularly because a key work (*Hetucakra,* "Wheel of Reason") is preserved only in the Tibetan and in the works of Naiyāyika opponents, and there seems to be a serious difference of opinion regarding its translation—a point on which I am not qualified to comment. While Dinnaga had no doubt introduced logical quantification, it seems to me necessary to grant that the quantification was based on a non-Aristotelian logic. In this connection, I would like to point to the last stanza of the *Hetucakra*. Matilal accepted that the standard negation does not fit Buddhist logic.[44]

My own reading is that Buddhist logic is quasi–truth-functional and that this quasi–truth-functionality of the underlying logic is closely related to the structure of time or the structure of the instant implicit in the Buddhist thesis of *paticca samuppada,* which, as the Buddha stated, is the key to the *dhamma*. Since I have amplified on this elsewhere, I will not go into the details here.

The Jaina logic[45] of *syādavāda* involves seven categories instead of Buddha's four and Sanjaya's five. The system is attributed to the commentator Bhadrabāhu. Jaina records and literature mention two Bhadrabāhus who lived about a thousand years apart. Between the two sects of the Jains there is no agreement as to the date of the later Bhadrabāhu, who may have lived as early as the fourth or as late as the fifth or sixth century, as his elaborate ten-limbed syllogism suggests, and if he really was the brother of the astronomer Varahamihira, whose work on astronomy is securely fixed at 498. The word *syat* may be translated as "may be," or as "perhaps," corresponding to *shāyad* in Hindustani. Hence, *syādavada* could be taken to mean "perhaps-ism" or "maybe-ism" or "discourse on the maybe." Uncertainty requires the making of judgments *(naya)*. The sevenfold judgments *(saptabhanginaya)* are: (1) *syadasti* (maybe it is), (2) *syatnasti* (maybe it is not), (3) *syadasti nasti ca* (maybe it is and is not), (4) *syadavaktavyah* (maybe it is inexpressible [= indeterminate]), (5) *syadasti ca avaktavyasca* (maybe it is and is indeterminate), (6) *syatnasti ca avaktavyasca* (may be it is not and is indeterminate), and (7) *syadasti nasti ca avaktavyasca* (maybe it is, is not, and is indeterminate). (According to some there is an eighth category, *vaktavasya avaktavasyaca*.)

Haldane's[46] interpretation of Bhadrabahu's[47] *syādavāda* is readily seen to correspond to the semantics of a three-valued logic. But Haldane achieves this by introducing a temporal separation between the assertions A and ~A. As everyone knows, in two-valued logic, $A \wedge \sim A \Rightarrow B$ for any B. The contradiction and resulting trivialization can be avoided by introducing a temporal separation between A and ~A as Haldane does: there is nothing paradoxical about Schrödinger's cat being alive now and dead a little while later.

I believe, however, that if temporal considerations are to be introduced, they may as well be introduced in a full-fledged way, so that one must then take into account also the differing notions of time and identity in the Buddhist and Jaina traditions.[48] If one does take into account the structure of time implicit in the Buddhist notion of instant and conditioned co-origination *(paticca samuppada),* then the natural logic to adopt is a quasi–truth-functional logic, and in this case one can meaningfully assert $A \wedge \sim A$ to hold *simultaneously,* that is, at one instant of time. I believe quasi–truth-functionality applies also to quantum logic,[49] and that this quasi–truth-functionality of logic is related to the empirical structure of time at the microphysical level, and particularly to the existence of microphysical time loops, but I will not elaborate further on the basis of microphysical time loops or my physical theory here.

The point of bringing in quantum logic is this: if one does eventually decide to appeal to the empirical, in support of logic, a two-valued logic need not be the automatic choice. Consider a meaningful but apparently contradictory proposition of the form "This pot is both red and black." The contradiction may be resolved by decomposing the proposition into the propositions "This part of the pot is red" and "That part of the pot is black." However, if the statements refer to the empirical, as we have now supposed, such a decomposition may end up referring to ever-smaller physical parts of the object. Thus, moving to atomic propositions may also drive one to the atomic domain in the physical world, where quantum mechanics certainly does apply. Thus, one might perhaps need to start with a quantum logic as the empirical basis of logic, so that *no* conclusion could be drawn from the statement that Schrödinger's cat is both dead and alive. (In two-valued logic, *any* conclusion could be drawn from this statement.) Specifically, the logic of the empirical world should not be regarded as a settled issue, solely on the basis of mundane experience.

In any case, there is no case for the "universality" of the logic underlying present-day mathematics and metamathematics. The alleged universality of two-valued logic fails across cultures, and it may well fail empirically. A two-valued logic may perhaps even fail as an industrial standard, for the internal logic of industrial capitalism drives technological innovation, and a two-valued logic does not apply to the formal semantics of parallel computing languages like OCCAM—which concern many parallel worlds (processors) in each of which a given statement may have independent truth values. Nor does a two-valued logic apply to quantum computers. (Quantum computers have been shown to be empirically viable, even if they cannot today be mass-marketed.)

If the logic underlying modern-day formalistic mathematics were to be changed, that would, of course, change also the valid theorems, as intuitionists demonstrated long ago. Hence, not only are the axioms of a formal mathematical theory arbitrary, but the allegedly universal part of mathematics—the relation of axioms to theorems through "proof"—is arbitrary since this notion of "proof" involves an arbitrary choice of logic. Logic is the key principle used to decide validity in formal mathematics, but it is not clear how this principle is to be fixed without bringing in either empirical or social and cultural considerations.

We see that the "universal" reason of the schoolmen was underpinned by the alleged authority of God to which the schoolmen indirectly laid claim. If this authority is denied, and Buddhists inevitably would deny it, there is nothing except practical and social authority that can be used to fix the logic used either within a formal theory or in a metamathematics that rejects an appeal to the empirical.

To summarize, all of present-day formal mathematics, in practice or in principle, depends on social and cultural authority, for whether or not a proposition is a mathematical theorem depends on Hilbert's notion of mathematical proof, and this notion of mathematical proof requires two-valued logic, which is not universal but depends on social and cultural authority. Thus, formal mathematics of the Hilbert-Bourbaki kind is entirely a social and cultural artifact. Proof or deduction provides only a social and cultural warrant for making cultural truth-assertions; it does not provide certain or secure knowledge.[50]

The Role of the Empirical

It is possible, of course, to argue that two-valued logic has social approval just because it is a matter of mundane empirical observation. But such an argument would hardly suit the twentieth-century Western vision of mathematics-as-proof, because once the empirical has been admitted at the base of mathematics, to decide logic itself, by what logic can it be excluded from mathematics proper? If the empirical world provides the basis of logic, why should the empirical be excluded from the process of logical inference? If the validity of *anumāna* is based on *pratyakṣa,* why should the *pratyakṣa* be excluded from valid *anumāna.*

Accepting the empirical may well make mathematics explicitly fallible, like physics. No one denies the fallibility of the empirical—as when one mistakes a rope for a snake or a snake for a rope. However, it seems to me manifest that social authority (e.g., that of Hilbert and Bourbaki) is *more* fallible than empirical observation. I regard the *pratyakṣa* as *more* reliable than *śabda* or authoritative testimony. Accordingly, I regard mathematics-as-calculation, based on the empirical, as *more* reliable, more secure, and more certain than mathematics-as-proof, which bypasses the empirical altogether.

To return to $2 + 2 = 4$, the particular case of $2 + 2 = 4$ still remains persuasive because, for example, two sheep when added to two sheep usually makes four sheep (although they may produce any number of sheep over a period of time). However, this involves an appeal to mundane human experience; it involves an appeal to the empirical, not the a priori.

Mundane experience may not be universal, but it is *more* universal than the a priori—there is less disagreement about mundane physical things than there is about metaphysics. Thus, the way to make mathematics more universal, and the way to evolve an East-West synthesis, is to accept the empirical in mathematics. The best route to universalization through an East-West synthesis is through everyday experience, through physics rather than metaphysics, through shared experience rather than shared acceptance of the same arbitrary social authority. Stable globalization needs *pramāṇa* rather than proof!

The argument above being abstract, a concrete example *(dṛṣtanta)* is in order. If mathematics is a social construction, then one can expect mathematics to change with changing technology and changing social circumstances. Can one point to instances of such change? Clearly that part of mathematics is most susceptible to change which is furthest away from the empirically manifest or *pratyakṣa.*

To bring this out, let us consider something for which there is no obvious empirical reference, such as division by zero. From the East-West point of view, zero is a particularly interesting case. We know that the notion of zero traveled from India to Europe via the algorismus texts, starting in tenth century, and that the epistemological assimilation of the zero required some five to six hundred years. As late as the late sixteenth century we find mathematicians in Europe worrying about the status of unity as a number, and the following question was still being used as a challenge problem: "Is unity a number?" The expected answer was that unity was not a number, but was the basis of number. With the changed social circumstance, those metaphysical concerns about the status of unity now merely serve to amuse us, and zero is now firmly regarded as a number, an integer. However, the nature of zero has changed.

Thus, Brahmagupta maintained that $0/0 = 0$. This is something that a modern-day mathematician will immediately regard as an error, for division by zero is not permitted. In present-day formal mathematics, zero is the additive identity; hence, for any number x, from the distributive law, $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$, so that $0 \cdot x = 0$. Thus zero cannot have a multiplicative inverse. Hence one cannot divide by zero, for division is nothing but the inverse of multiplication. Hence, Datta and Singh assert that Brahmagupta was mistaken.[51] At a conference on *śūnya,*[52] almost all the participants agreed with this perception by Datta and Singh. (I was the exception.) This goes to show the extent of acculturation, but not, of course, the universal validity of the belief. The proof above of the illegitimacy of division by zero tacitly assumes that the numbers in question must form a field, but, as we have already seen, this is not the case for numbers on a computer, where the distributive law, used in the proof above, fails.

As a matter of fact, there are, even in current mathematics, common situations where $0/0 = 0$ may implicitly be used as part of the arithmetic of extended real numbers. Thus, consider the Lebesgue integral:

$$(1) \quad \int_0^1 \frac{1}{\sqrt{x}}\, dx = 1$$

The integrand is ill behaved only when $x = 0$, when the denominator becomes zero. Since the integral is a Lebesgue integral rather than a Riemann integral, we do not omit zero from the region of integration, but appeal to the rules of the extended real number system,[53] which admits the additional symbols ∞, $-\infty$.

Now, either the limit

$$\lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty$$

or the corresponding *unwritten* convention

(2) $1/0 = \infty$

allows us to regard the integrand as

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \neq 0 \\ \infty, & x = 0 \end{cases}.$$

However, the integrand is infinite only at a single point; that is, it is infinite only on a set of Lebesgue measure zero. Hence, we appeal to the standard convention, used in the theory of the Lebesgue integral, that

(3) $0 \cdot \infty = 0$.[54]

We see that (1), (2), and (3) together amount to saying that $0/0 = 0$. I would emphasize that the convention (3) $0 \cdot \infty = 0$ is a very important convention, for one cannot do modern-day probability theory or statistics without it; a statement that is true with probability one, that is, true except on a set of probability zero, is said to be true almost everywhere, and occurs almost everywhere in current probability theory. Thus, $0/0 = 0$ is certainly not a convention every use of which is *necessarily* incorrect. This was presumably believed to be so in 1935 by Datta and Singh, but we now have good reasons for admitting the convention, at least in some situations—reasons relating both to mathematical practice and to computer arithmetic. But can one make $0/0 = 0$ a universal rule? That depends, in the first place, on what one means by zero.

Under different social and cultural circumstances, zero was regarded differently. As I have argued elsewhere,[55] in Brahmagupta's case, *śūnya* or zero is not the additive identity in a field, but refers to the non-representable, in line with the meaning given to it in the *śūnyavāda* of Nāgārjuna. With calculations involving a representable, hence a finite, set of numbers, such non-representable numbers are bound to arise, and some rule is needed to handle these cases. Brahmagupta's rule should be read as

$$nr/nr = nr,$$

where $nr =$ non-representable.

We see that changed social circumstances have transformed the notion of zero, but further social changes could change this notion further. As observed above, computers can represent only a finite set of numbers. Hence, exactly this problem of dealing with non-representable numbers arises in computing. Here, too, we have a situation very similar to $nr/nr = nr$, as can be seen by writing and executing the accompanying short C program (fig. 4). In accordance with Western

```
/*Program name: sunya.c */
/*Function: To show how a computer handles non-
representable numbers according to the IEEE standard */

#include <stdio.h>
#include <conio.h>
#include <values.h>
main()
{
      float a, b, c;
      a =  MAXFLOAT;
      b =  MINFLOAT;
      printf ("a = %e, b = %e", a, b);
      getch();
/*Now try putting in values of a, and b, larger than
MAXFLOAT or values of b smaller than MINFLOAT */
      printf ("\n\n Enter a = ");
      scanf ("%f", &a);
      printf ("a = %f", a);
      printf ("\n Enter b = ");
      scanf ("%f", &b);
      printf ("b = %f", b);
      c = a/b;
      printf ("%e/%e = %e", a, b, c);
      /*printf ("%f/%f = %f", a, b, c);*/ /*uncomment*/
      getch();
      return 0;
}
```

Program Input and Output:

```
a = 3.37000e+38
b = 8.43000e-37
Enter a = 1e40
a = +INF
Enter b = -1e40
b = -INF
Floating point error: Domain
```

Fig. 4. How a computer handles the non-representable. The program above illustrates how a computer handles non-representable numbers. 1e40 denotes the number 10^{40}, while -1e40 denotes the number -10^{40}. Instead of saying that $a/b = -1$, the computer states that there has been an error.

mathematical sensibilities, the IEEE standard, however, permits a few different *types* of non-representables. Anything smaller in absolute value than 1.40130e − 45 is non-representable and is represented by zero. Anything larger than 3.37e + 38 is non-representable but is represented by +INF, while anything smaller than −3.37e − 38 is represented by −INF. Even though the associative and distributive laws fail for numbers on a computer, in accordance with prevalent Western mathematical conventions, the IEEE standard specifies that arithmetic operations involving non-representables, such as 0/0 *always* lead to an undefined result, which is treated as an error. (This is not the full story, and there are other kinds of non-representables. Indeed, by uncommenting the line marked "uncomment," that is, by removing the first pair of `/*` and `*/` in that line, and providing the inputs a = 2.0e − 45 and b = 4.0e − 45, one can actually make the computer print out the statement 0.00000/0.00000 = 0.00000! But this is not something that needs to be taken seriously.)

How satisfactory are the IEEE specifications that 0/0 = 0 *always* is an error? If we look upon this as a practical matter of making efficient calculations, then a universal rule of the kind that one has in present-day computing is *not* the most efficient. For example, in a practical situation, even if something is treated as non-representable, we might yet know that it is the *same* non-representable as one that was previously encountered. In that case, we may even want to apply the cancellation law to zero! We might want to say

$$\frac{2 \cdot 10^{46}}{4 \cdot 10^{46}} = 1/2.$$

But this is a statement that the IEEE standard regards as erroneous for floats (real numbers represented in single precision), as the accompanying C program shows. According to that standard, the correct statement is:

$$\frac{2 \cdot 10^{46}}{4 \cdot 10^{46}} = \text{Floating-point error.}$$

Accordingly, the computer treats the attempt to carry out the calculation above as erroneous, although anyone can see what the valid answer is. Thus, the attempt to eliminate one kind of absurdity (that might arise out of a wrong use of 0/0 = 0) leads to another kind of absurdity.

A machine cannot discriminate between a "legitimate" use of 0/0 = 0 and an "illegitimate" use: it cannot easily handle exceptional situations; it needs a universal rule, and this universal rule may lead to other absurdities. Although the IEEE has regarded the latter absurdity as more acceptable, this could change with circumstances. The conventions may change not only with who lays down the standard, but also with who performs the calculation: for human arithmetic, as distinct from machine arithmetic, we may use rules that permit exceptions. This is exactly how Bhaskara II interprets Brahmagupta's rule while computing the value of $x(= 44)$, given that

$$\frac{x \cdot 0 + \frac{x \cdot 0}{2}}{0} = 63.$$[56]

This suggests that when we go beyond the empirical, the "universal" may lie, as in a physical theory, in what Poincaré called "convenience." This criterion of "convenience" can have profound consequences as in the case of the theory of relativity: the constancy of the speed of light is not an empirical fact (although elementary physics texts usually misrepresent it as such); Poincaré defined the speed of light as a constant as a matter of "convenience." I see this criterion of "convenience" as more modest than the criterion of beauty, which seeks to globalize a local sense of aesthetics.

History of the Calculus

If mathematics is a social construct, which changes with changing social circumstances, then the question is: how should one teach mathematics today? Admitting the role of technology in shaping mathematics, accepting that the computer is going to play an increasingly important role in the future, and admitting that formal mathematics is not quite suited to computers, the conclusion seems to be forced that a different type of mathematics should be taught. The calculus is at the core of many numerical computations, but can one do the calculus at all without real numbers? An alternative mathematical epistemology could be invented ab initio. Or one could fall back on the alternative epistemology of mathematics in India, as described in the *Yuktibhāṣā.* This alternative epistemology provided the natural soil in which the calculus grew. Recognizing the existence of this alternative epistemology of mathematics requires, however, an alternative account of the history of mathematics. This is an illustration of the general maxim that the history of mathematics has profoundly influenced its philosophy, so that to change the philosophy of mathematics one must also revise its history. A condensed account of the suggested revision follows.

According to the Western history of the calculus, the calculus was the invention of Leibniz and Newton, particularly Newton, who used it to formulate his "laws" of physics. In a series of papers, I have pointed out that this narrative needs to be significantly changed for several reasons.

1. The key result of the calculus, attributed variously to James Gregory,[57] Newton, and to Newton's student Brook Taylor,[58] is the infinite-series expansion commonly known today as the Taylor's series expansion. This infinite-series expansion is found in India a few centuries before Newton in the work of Madhava of Sangamagrama and in later works like Nilkantha's *TantraSangraha* (1501 C.E.), Jyeshtadeva's *YuktiBhāṣa* ("Discourse on rationale") (ca. 1530 C.E.),[59] the *TantraSangrahaVyākhya,* the *YuktiDīpikā,* the *Kriyākramakari,* the *KaraṇaPadhati,* and other such widely distributed and still extant works of what has been called the Kerala school of mathematics and astronomy.

निहत्य चापवर्गेण चापं तत्तत् फलानि च ।
हरेत समूलयुग्वर्गैस्त्रिज्यावर्गाहतैः क्रमात् ॥
चॉपं फलानिं चाधोऽधौ न्यस्योपर्युपरि त्यजेत् ।
जीवाप्त्यै...

This key passage may be translated as follows:

> Multiply the arc by the square of the arc, and repeat [any number of times]. Divide by the product of the square of the radius times the square of successive even numbers increased by that number [multiplication being repeated the same number of times]. Place the arc and the results so obtained one below the other and subtract each from the one above. These together give the *jīvā*. . . .

Jīvā relates to the sine function. Etymologically, the term "sine" derives from *sinus* (fold), a Latin translation of the Arabic *jaib* (opening for the collar in a gown), which is a misreading of the Arabic term *jībā* (both terms are written as *jb*, omitting the vowels). Mathematically, however, as is well known, *jīvā* and *śara*, like the sine and cosine of Clavius' sine tables (as their very title shows),[60] were not the modern sine and cosine but these quantities multiplied by the radius r of a standard circle. The *jīvā* corresponds to $r \sin \theta$, while the *śara* corresponds to $r(1 - \cos \theta)$.

In current mathematical terminology, this passage says the following: let r denote the radius of the circle, let s denote the arc, and let t_n denote the nth expression obtained by applying the rule cited above. The rule requires us to calculate as follows.

(1) Numerator: multiply the arc s by its square s^2, this multiplication being repeated n times to obtain

$$s \cdot \prod_{1}^{n} s^2.$$

(2) Denominator: Multiply the square of the radius, r^2, by $[(2k)^2 + 2k]$ ("square of successive even numbers increased by that number") for successive values of k, repeating this product n times to obtain

$$\prod_{k=1}^{n} r^2[(2k)^2 + 2k].$$

Thus, the nth iterate is obtained by

$$t_n = \frac{s^{2n} \cdot s}{(2^2 + 2) \cdot (4^2 + 4) \cdot \ldots \cdot [(2n)^2 + 2n] \cdot r^{2n}}.$$

The rule further says:

$$jīvā = (s - t_1) + (t_2 - t_3) + (t_4 - t_5) + \cdots$$

Substituting

(1) $jīvā \equiv r \sin \theta$

(2) $s = r\theta$, so that $s^{2n+1}/r^{2n} = r\theta^{2n+1}$,

and noticing that

(3) $[(2k)^2 + 2k] = 2k \cdot (2k+1)$, so that

(4) $(2^2+2)(4^2+4)\cdots[(2n)^2+2n] = (2n+1)!$

and cancelling r from both sides, we see that this is entirely equivalent to the well-known expression

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

This verse is followed by a verse describing an efficient numerical procedure for evaluating the polynomial.[61] The existence of these verses has been known to Western specialists for nearly two hundred years, and is today acknowledged in some Western texts on the history of mathematics, like those of Jushkevich,[62] Katz,[63] and others.

In current mathematical terminology, the key step in the *Yuktibhāṣā* rationale for the series above is that

$$(1) \quad \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{1}^{n} i^k = \frac{1}{k+1}, \quad k = 1, 2, 3, \ldots$$

in the sense that the remaining terms are numerically insignificant, for large enough n.

2. A relevant epistemological question is this: did Newton at all understand the result he is alleged to have invented? Did Newton have the wherewithal, the necessary mathematical resources, to understand infinite series? As is well known, Cavalieri in 1635 stated the formula above as what was later termed a conjecture. Wallis, too, simply stated the result above, without any proof.[64] Fermat tried to derive the key result above from a result on figurate numbers, while Pascal used the famous "Pascal's" triangle,[65] long known in India and China. Although Newton followed Wallis, he had no proof either,[66] and neither did Leibniz, who followed Pascal. Neither Newton nor any other mathematician in Europe had the mathematical wherewithal to understand the calculus for another two centuries, until the development of the real number system by Dedekind.

3. The next question naturally is this: if Newton and Leibniz did not quite understand the calculus, how did they invent it? In the amplified version of the usual narrative, how did Galileo, Cavalieri, Fermat, Pascal, and Roberval et al. all contribute to the invention of a mathematical procedure they could not quite have understood? The frontiers of a discipline are usually foggy, but here we are talking of a gap that is typically 250 years.

4. Clearly a more natural hypothesis to adopt is that the calculus was not invented in Europe, but was imported, and that the calculus took nearly as long to assimilate as did the zero. Since authoritative Western histories of mathematics are

replete with wild claims of transmission from Greece, an appropriate standard is needed for the evidence for transmission. I have suggested that we follow the current legal standard of evidence, by establishing (1) motivation, (2) opportunity, (3) documentary evidence, and (4) circumstantial evidence.

Motivation (a). Europe had a strong motivation to import mathematical and astronomical knowledge in the sixteenth and seventeenth centuries, because mathematics and astronomy were widely regarded as holding the key to navigation, which was the route to prosperity and hence the critical technology of the times. As is now widely known, Europe did not have a reliable technique of navigation, and European governments kept offering huge prizes for this purpose from the sixteenth century until the eighteenth. Indeed, the French Royal Academy, the Royal Society of London, and so forth were started in this way in an attempt to develop the astronomical and mathematical procedures needed for a reliable navigational technique.

The first navigational problem concerned latitude: right from Vasco da Gama, Europeans attempted to learn the Indo-Arabic techniques of determining latitude through instruments like the Kamāl. The Indo-Arabic technique of determining latitude in daytime assumed a good calendar, and this led to the Gregorian calendar reform. As a student and correspondent of Pedro Nunes, Clavius presumably understood that reforming the calendar and changing the date of Easter was critical to the navigational problem of determining latitude from the observation of solar altitude at noon, as described in widely distributed Indian mathematical-astronomical texts and calendrical manuals.

Opportunity. On the other hand, right from the sixteenth century there was ample opportunity for Europeans to collect Indian mathematical-astronomical and calendrical texts. The Jesuits were in India, with their strongest center being Cochin, from where a copy of the *Tantrasangraha* or *Yuktibhāṣā* could easily have been procured. Each Jesuit was expected to know the local language, and Alexander Valignano declared that it was more important for the Jesuits to know the local language than to learn philosophy. They could hardly have functioned without a knowledge of the local calendar and days of festivity. One of the earliest Jesuit colleges was at Cochin, and it typically had an average of about seventy Jesuits during the period 1580–1660. Prior to this period, printing presses had already been started in languages like Malayalam and Tamil, and Malayalam was being taught at the Cochin college at the latest by 1590.

Documentary Evidence. Moreover, the Jesuits were systematically collecting and translating local texts and sending them back to Europe. In particular, Christoph Clavius, head of the Gregorian Calendar Reform Committee, changed the mathematics syllabus of the Collegio Romano, to correct the Jesuit ignorance of mathematics, and from the first batch of mathematically trained Jesuits he sent Matteo Ricci to Cochin to understand the available texts in India on the calendar and the length of the year.[67]

Motivation (b). Pedro Nunes was also concerned with loxodromic curves, the key aspect of Mercator's navigational charts, which involved a problem equivalent to the fundamental theorem of calculus. Pedro Nunes obtained his loxodromic curves using sine tables, which tables were later corrected by Christoph Clavius and Simon Stevin. Thus, precise sine values were a key concern of European astronomers and navigational theorists of the time. The infinite-series expansion as used by Madhava to calculate high-precision sine values, the coefficients used for efficient numerical calculation of these values, and the twenty-four values themselves were incorporated in a single sloka each, the last two found also in the widely distributed calendrical manuals like *Karaṇapadhati.*

Motivation (c). Europeans could not use Indo-Arabic techniques of longitude determination because of a goof-up about the size of the earth. Columbus, to promote the financing of his project, downgraded the earlier, accurate Indo-Arabic estimates of the size of the earth by 40 percent. But this size entered as a key parameter in the Indo-Arabic techniques. Nevertheless, Europeans remained interested in the Indo-Arabic techniques of longitude determination, and when the French Royal Academy ultimately developed a method to determine longitude on land, it was a slight improvement of the technique of eclipses mentioned in the texts of Bhaskara I and in the tome of al Biruni.

Circumstantial Evidence. Once in Europe, the imported mathematical techniques could easily have been diffused, and there is circumstantial evidence that many contemporary mathematicians knew something of the material in Indian texts. For example, Clavius' competitor and critic Julian Scaliger introduced the Julian day-number system, essentially the *ahārgaṇa* system of numbering days followed in Indian astronomy since Aryabhata. Galileo's access to Jesuit sources is well documented, as is that of Gregory and of Wallis. Cavalieri was Galileo's student, and Gregory does not claim originality for his series. Marin Mersenne was a clearinghouse for mathematical information, and his correspondence records his interest in the knowledge of Brahmins and "Indicos." Fermat, Pascal, and Roberval were all in touch with him, and part of his discussion circle. There is other circumstantial evidence to connect Fermat to Indian mathematical texts, for instance his famous challenge problem to European mathematicians, and particularly to Wallis, involves a solved problem in Bhaskara's *Beejgaṇita.*[68] The "Julian" day-number, "Fermat's" challenge problem, and "Pascal's" triangle cover only some of the circumstantial evidence of the inflow of mathematical and astronomical knowledge into Europe of that period, but I will not examine more details here, since I regard what has been stated above as adequate to make a strong case for the transmission of the calculus from India to Europe in the sixteenth and seventeenth centuries.

Mathematics Education

To jump from the past to the future: what bearing do these concerns have on current mathematics education? In the light of the revised history of the calculus, in the light of the argument that mathematics is a social construction that is likely to change with changing technology, especially the widespread use of computers, how should mathematics and calculus be taught today?

In accordance with the principle that *phylogeny* is *ontogeny,* the natural way to learn the subject is to retrace its ontogenesis. The current way of teaching the calculus retraces the ontogenesis of the calculus in Europe. The calculus is first taught as an intuitive and unclearly understood thing that is nevertheless indispensable for practical purposes. After at least a couple of years (representing the gap of a couple of centuries in Europe), one teaches the real-number system, the elements of mathematical analysis, and the Riemann integral, finally leading to a proof of the so-called Taylor's theorem, the classical version of the fundamental theorem of calculus, and Peano's existence theorem for the solution of differential equations, and so forth. Numerical analysis, and discretization, is typically expected to come *after* this. Since pedagogy follows the (perceived) ontogeny, the revised ontogenesis suggests a revised way to teach mathematics. The "numerical calculus" of the *Yuktibhāṣā*, as distinct from both calculus and analysis, can be taught directly as a technique of computation, using floating-point numbers and an empirical rationale.

A similar conclusion follows from the argument that formal mathematics is a social construction, likely to change with technology. The computer has simplified complex calculations enormously, and has thus encouraged the view of mathematics-as-calculation. By encouraging the idea of mathematics-as-calculation, computer technology has already created sharp conflicts with Western mathematical orthodoxy, and its theological orientation toward mathematics-as-proof. Ideally one is expected to prove a convergence theorem for an algorithm before writing a computer program for it. Ideally one should even prove the program that one uses: of what value is a computer-aided proof of the four-color theorem if the program used in the proof cannot itself be proved? This requirement of proof is rarely respected in practice. Few people who use computers (e.g., physicists and engineers) have enough mathematical training to provide proofs of this kind. Even if they have, the required proofs may simply not be available, as in the case, mentioned earlier, of stochastic differential equations driven by Lévy motion. A practical requirement must be met here and now. For a practical requirement, one generally cannot wait for as long as one may be ready to wait to demonstrate the validity of an eternal truth.

Both arguments suggest that it is time to revise the mathematics syllabus outlined by Plato.

1. Mathematics-as-calculation should be taught for its practical value, at the elementary and intermediate levels. This applies especially to the calculus, given its revised ontogenesis and given its implementation on computers.

2. Mathematics must be taught as empirically based and fallible. Thus, certainly, the question no longer is *what* is the value of $1 - 1 + 1 - 1 + 1 - 1 \ldots$, nor is it any longer *how* should one define $1 - 1 + 1 - 1 + 1 - 1 \ldots$ so as to lead to a theory most acceptable to authoritative mathematicians. Rather, the question is this: are there methods of summing this series that are empirically useful? Hence, a technique of calculation, for example $1 - 1 + 1 - 1 + 1 - 1 \cdots = 1/2$, could be acceptable if it is of practical value, like an engineering technique, or can be empirically validated, like a physical theory or in conjunction with a physical theory. A given technique of calculation may be fallible and may not work in another case: for

example, the standard technique of extracting a finite value from a divergent integral, as used in renormalization in quantum field theory, does not work with shock waves. While one need have no qualms about nonuniversality, naturally the most convenient conventions will be those that are most widely applicable.

3. On the other hand, I feel that Proclus did have a point: that at least at an elementary level, mathematics-as-proof does afford a certain aesthetic satisfaction, even if mathematics-as-proof does not fulfill the original promise of providing secure knowledge. Thus, I feel that the teaching of mathematics-as-proof, like the teaching of music or some other art form, ought not to be discontinued altogether, but should be an optional matter, which could be taken up, especially at higher levels, by those interested in it.

Notes

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1 – *Proclus: A Commentary on the First Book of Euclid's Elements,* trans. Glenn R. Morrow (Princeton: Princeton University Press, 1970), p. 3.

2 – Plato *Republic* VII.526, in *The Republic of Plato,* trans. into English, with an analysis and notes by John Llewelyn Davies and David James Vaughan (1912; London: Macmillan), p. 240. Benjamin Jowett's translation reads: "if geometry compels us to view being it concerns us; if becoming only, it does not concern us" (*The Dialogues of Plato,* trans. Benjamin Jowett [Chicago: Encyclopaedia Britannica, 1996], p. 394).

3 – Rescher gives a very detailed account of the Aristotelian and Diodorean temporalized modalities and how these were interpreted by medieval European commentators; see N. Rescher, "Truth and Necessity in Temporal Perspective," in Richard M. Gale, ed., *The Philosophy of Time: A Collection of Essays* (London: Macmillan, 1962), pp. 183–220. Al Ghazālī, in his *Tahāfut al Falāsifā,* opposed the philosophers and rational theologians of Islam exactly on the grounds that any necessary component of the empirical world would restrict the powers of God, who continuously created the world; however, even al Ghazālī did not deny that God was compelled by (Aristotelian) logical necessity.

4 – The notion of "truth" has, of course, had a variety of meanings in mathematics. Paul Ernest adopts the interesting terminology of "truth_1" for the traditional European notion of mathematical truth, akin to the naive realism (but without its empirical basis) prevalent until around the mid-nineteenth century; "truth_2" for Tarski's notion of satisfiability, or true in a possible world, which presumably originated with Hilbert's work on geometry; and "truth_3" for the notion of logical validity, which roughly corresponds to "true" in all possible worlds. See

Paul Ernest, *Social Constructivism as a Philosophy of Mathematics* (State University of New York Press, 1998), chap. 1. While I will not adopt this terminology explicitly, I hope the sense in which the word "true" is used will be clear from the context.

5 – For example, Elliot Mendelson, *Introduction to Mathematical Logic* (Princeton: Van Nostrand, 1964), p. 29.

6 – S. N. Sen and A. K. Bag, *The Sulbasūtras* (New Delhi: Indian National Science Academy, 1983).

7 – Kripa Shankar Shukla, ed., in collaboration with K. V. Sarma, *Āryabhatīya of Āryabhata* (New Delhi: Indian National Science Academy, 1976).

8 – *YuktiBhāṣā,* pt. 1, ed. with notes by Ramavarma (Maru) Thampuran and A. R. Akhileṣwara Aiyer (Trichur: Mangalodayam Ltd., 1123 Malayalam Era; 1948 C.E.), unpublished English translation by K. V. Sarma.

9 – For the double set of quotation marks, see C. K. Raju, "How Should 'Euclidean' Geometry be Taught" (paper presented at the International Workshop on History of Science: Implications for Science Education, TIFR, Bombay, February 1999; to appear in their Proceedings). Briefly, there are two reasons, respectively, for the two sets of quotation marks. (1) Although the result was clearly known prior to Pythagoras, and Proclus regards the attribution to Pythagoras as a "rumor," the attribution to Pythagoras has been sustained on the grounds that mathematics ought not to involve the empirical. (2) However, if that be the case, and we adopt Hilbert's synthetic approach for consistency, then it has not been widely noticed that proposition 1.47 of the *Elements* is no longer valid or syntactically acceptable, for it asserts "equality" in the sense of equal areas, and area, like length, is a metric notion, not available in Hilbert's synthetic approach, which substitutes Proclus' equality with congruence. This substitution does not apply to 1.47, since the areas involved are noncongruent.

10 – Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (1908; New York: Dover Publications, 1956), vol. 1.

11 – While the Mut'azilāh retained the Neoplatonic concern for equity, Western rational theologians abandoned it, for the church had long before cursed Origen and the Neoplatonists precisely for their advocacy of equity. See Raju, "How Should 'Euclidean' Geometry be Taught."

12 – In Christian rational theology, the empirical world had to be contingent, since a necessary proposition was regarded as a proposition that had to be true for all time, or at least for all future time (Rescher, "Truth and Necessity in Temporal Perspective"). But, a world that existed for all time past, or all time future, would go against the doctrine of creation and apocalypse.

13 – Proclus, *Proclus: A Commentary,* p. 29.

14 – Ibid., p. 37: "as Plato also remarks: 'If you take a person to a diagram,' he says, 'then you can show most clearly that learning is recollection.'"

15 – David Hilbert, *The Foundations of Geometry*, trans. E. J. Townsend (Chicago: Open Court, 1902).

16 – Bertrand Russell, *An Essay on the Foundations of Geometry* (1897; 1908; 1956; London and New York: Routledge, 1996).

17 – School Mathematics Study Group, *Geometry* (New Haven: Yale University Press, 1961).

18 – Gödel's attack on Hilbert's program concerned Hilbert's belief that theorems could be *mechanically* derived from axioms. Gödel's theorems do not challenge the notion of proof, which remains mechanical. That is, although it may be impossible to *generate* mechanically or recursively the proof of all theorems pertaining to the natural numbers, given a fully written-out proof, it is, in principle, possible to *check* its correctness *mechanically.*

19 – Arthur Schopenhauer, *Die Welt als Wille,* 2d ed. (1844), p. 130; cited in Heath, *The Thirteen Books of Euclid's Elements,* p. 227.

20 – G. D. Birkhoff, "A Set of Postulates for Plane Geometry (Based on Scale and Protractor)," *Annals of Mathematics* 33 (1932). For an elementary elaboration of the difference between the various types of geometry, see Edwin E. Moise, *Elementary Geometry from an Advanced Standpoint* (Reading, Massachusetts: Addison-Wesley, 1968).

21 – Plato *Republic* 533, in Davies and Vaughan, *The Republic of Plato,* p. 248. Jowett's translation reads: "—geometry and the like—they only dream about being, but never can they behold the waking reality so long as they leave the hypotheses which they use unexamined, and are unable to give an account of them. For when a man knows not his own first principle, and when the conclusion and intermediate step are also constructed out of he knows not what, how can he imagine that such a fabric of convention can ever become science?" (Jowett, *The Dialogues of Plato,* p. 397).

22 – Proclus, *Proclus: A Commentary,* p. 26.

23 – Ibid., p. 33.

24 – Ibid., p. 24.

25 – Plato *Republic* 527, in Davies and Vaughan, *The Republic of Plato,* p. 240. Jowett's translation reads: "They have in view practice only, and are always speaking, in a narrow and ridiculous manner, of squaring and extending and applying and the like—they confuse the necessities of geometry with those of daily life; whereas knowledge is the real object of the whole science" (Jowett, *The Dialogues of Plato,* p. 394).

26 – Suzan Rose Benedict, *A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning* (Ph.D. diss., University of Michigan, April 1914; Concord, New Hampshire: Rumford Press, 1916).

27 – The victory of algorismus over abacus was depicted by a smiling Boethius using Indian numerals, and a glum Pythagoras to whom the abacus technique was attributed. This picture first appeared in the *Margarita Philosophica* of Gregor Reisch in 1503, and is reproduced, for example, in Karl Menninger, *Number Words and Number Symbols: A Cultural History of Numbers,* trans. Paul Broneer (Cambridge: MIT Press, 1970), p. 350.

28 – C. K. Raju and Dennis Almeida (Aryabhata Group), "The Transmission of the Calculus from Kerala to Europe, Part I: Motivation and Opportunity" and "... Part II: Documentary and Circumstantial Evidence" (paper presented at the Aryabhata Conference, Trivandrum, January 2000).

29 – More details, quotations, etc. may be found in C. K. Raju, "Kamāl or Rapalagai" (paper presented at the Indo-Portuguese Conference on History, Indian National Science Academy, December 1998; to appear in their Proceedings).

30 – "Com tudo não me parece que sera impossivel saberse, mas has de ser por via d'algum mouro honorado ou brahmane muito intelligente que saiba as cronicas dos tiempos, dos quais eu procurarei saber tudo" (letter by Matteo Ricci to Petri Maffei on 1 December 1581, in *Goa* 38 I, ff. 129r–130v, corrected and reproduced in *Documenta Indica* 12 : 472–477 at p. 474. Also reproduced in Tacchi Venturi, *Matteo Ricci S.I., Le Lettre Dalla Cina 1580–1610,* vol. 2 [Macerata, 1613]).

31 – *The Principal Works of Simon Stevin,* vol. 3, *Astronomy and Navigation,* ed. A. Pannekoek and Ernst Crone (Amsterdam: Swets and Seitlinger, 1961).

32 – Christophori Clavii Bambergensis, *Tabulae Sinuum, Tangentium et Secantium ad partes radij 10,000,000....* (Ioannis Albini, 1607).

33 – The proposition is: to construct an equilateral triangle on a given finite line segment.

34 – Ubiratan D'Ambrosio, *Socio-Cultural Bases for Mathematics Education* (Unicamp, 1985).

35 – ANSI/IEEE Standard 754–1985.

36 – Incidentally, there is the practical observation that this notion of implication is counterintuitive: students have a hard time grasping that $A \Rightarrow B$ is true provided only that A is false. This suggests that the notion of implication in two-valued logic, far from being universal, is not even the same as the notion of implication in natural language.

37 – C. K. Raju, "The Mathematical Epistemology of *Śūnya*" (summary of interventions at the Seminar on the Concept of *Śūnya*, Indian National Science Academy and Indira Gandhi National Centre for the Arts, New Delhi, February 1997); C. K. Raju, "Mathematics and Culture," in *History, Culture and Truth, Essays Presented to D. P. Chattopadhyaya,* ed. Daya Krishna and K. Satchidananda Murty (New Delhi: Kalki Prakash, 1999), pp. 179–193, and reprinted

in *Philosophy of Mathematics Education* 11, available at http://www.ex.ac.uk/~PErnest/pome11/art18.html; C. K. Raju, "Some Remarks on Ontology and Logic in Buddhism, Jainism and Quantum Mechanics" (Invited talk at the conference on Science et engagement ontologique, Barbizon, October 1999).

38 – Prior to the Buddha, a different logic may have been prevalent, as Barua argues. In Barua's view Sanjaya Belatthaputta used a fivefold negation: *evam pi me no, tath ti pi me no, annatha ti pi me no, iti ti pi me no, no ti ti pe me no* (Benimadhab Barua, *A History of Pre-Buddhistic Indian Philosophy* [Calcutta, 1921; reprint, Delhi: Motilal Banarsidass, 1970]).

39 – Maurice Walshe, trans., *The Long Discourses of the Buddha: A Translation of the Dīgha Nikāya* (Boston: Wisdom Publications, 1995), p. 541 n. 62 (to Sutta 1).

40 – Walshe, *The Long Discourses of the Buddha,* pp. 78–79.

41 – *Dīgha Nikāya,* in Walshe, *The Long Discourses of the Buddha,* pp. 80–81.

42 – *Mūlamādhyamakakārikā* 18.8, Sanskrit text and English trans. David J. Kalupahana, in *Nagarjuna: The Philosophy of the Middle Way* (New York: State University of New York Press, 1986), p. 269.

43 – D. Chatterji, trans., "Hetucakranirnaya," *Indian Historical Quarterly* 9 (1933): 511–514; reproduced in full in R.S.Y. Chi, *Buddhist Formal Logic* (London: The Royal Asiatic Society, 1969; reprint, Delhi: Motilal Banarsidass, 1984). Chi objects to the exposition of Vidyabhushan (cited note 45 below).

44 – B. K. Matilal, *Logic, Language, and Reality* (Delhi: Motilal Banarsidass, 1985), p. 146: "My own feeling is that to make sense of the use of negation in Buddhist philosophy in general, one needs to venture outside the perspective of the standard notion of negation." See also, H. Herzberger, "Double Negation in Buddhist Logic," *Journal of Indian Philosophy* 3 (1975): 1–16.

45 – Satis Chandra Vidyabhusana, *A History of Indian Logic: Ancient, Mediaeval, and Modern Schools* (Calcutta, 1921).

46 – J.B.S. Haldane, "The Syadvada System of Predication," *Sankhya: Indian Journal of Statistics* 18 (1957): 195; reproduced in D. P. Chattopadhyaya, *History of Science and Technology in Ancient India* (Calcutta: Firma KLM, 1991), as vol. 2, *Formation of the Theoretical Fundamentals of Natural Science,* Appendix 4, pp. 417–432.

47 – Vidyabhusana, *A History of Indian Logic.* Neither Jaina nor Buddhist records tell us which Bhadrabahu was associated with *syādvāda.* I am inclined to think it was Bhadrabahu the junior, a contemporary of Dinnaga, and not the senior.

48 – C. K. Raju, "Time in Indian and Western Traditions, and Time in Physics," in *Mathematics, Astronomy and Biology in Indian Tradition,* ed. D. P. Chattopadhyaya and Ravinder Kumar (New Delhi: PHISPC, 1995), pp. 56–93; C. K. Raju, "Kāla and Dik," forthcoming in P. K. Sen, ed. (New Delhi: Project of History of Indian Science, Philosophy, and Culture. Vol. XII).

49 – C. K. Raju, *Time: Towards a Consistent Theory* (Dordrecht: Kluwer Academic Publishers, 1994), chap. 6b.

50 – Isn't socially approved knowledge the best that one can aspire to? That depends upon the nature of the society in question. Does social authority refer to unanimity or even to a democratically evolved consensus? As I have argued elsewhere, in industrial-capitalist societies, for economic reasons, the social authority for scientific knowledge necessarily rests in certain specialists, and the social conferment of authority on these specialists often fully reflects the evils of these societies. Further, these specialists being under pressure to conform, the agreement of many specialists is hardly a guarantee of secure knowledge. Thus, mathematical knowledge in capitalist societies is exactly as insecure as the technology that arises from the capitalist way of getting quick practical results with the least resources (C. K. Raju, "Mathematics and Culture").

51 – Bibhutibhusan Datta and Avadhesh Narayan Singh, *History of Hindu Mathematics: A Source Book,* pts. 1 and 2 ([1935]; Bombay: Asia Publishing House, 1962), p. 245: "Brahmagupta has made the incorrect statement that $0/0 = 0$."

52 – Seminar on the Concept of *Śūnya,* Indian National Science Academy and Indira Gandhi National Centre for the Arts, New Delhi, February 1997.

53 – Walter Rudin, *Real and Complex Analysis* (New Delhi: Tata McGraw Hill, 1966), pp. 18–19.

54 – Ibid.

55 – C. K. Raju, "The Mathematical Epistemology of *Śūnya*" (cited note 37 above).

56 – Datta and Singh, *History of Hindu Mathematics,* p. 245. There is a misprint there about the value of x.

57 – In a letter of 15 February 1671 to John Collins, Gregory had supplied Collins with seven-power series around 0, for $\arctan\theta$, $\tan\theta$, $\sec\theta$, $\log\sec\theta$, etc. (Herbert Westren Turnbull, *James Gregory Tercentenary Memorial Volume* [London: Published for the Royal Society of Edinburgh by G. Bell and Sons, 1939]). Gregory's series, however, contained some minor errors in the calculation of the coefficient of the fifth-order term in the expansion.

58 – The Taylor Theorem appeared as Proposition 7, Corollary 2, in Brook Taylor, *Methodus Incrementorum directa et inversa* (London, 1715, 1717). A translation is in L. Feigenbaum, "Brook Taylor's 'Methodus Incrementorum': A Translation with Mathematical and Historical Commentary" (Ph.D. diss., Yale University, 1981). Apart from Newton, the series was anticipated by James Gregory (L. Feigenbaum, "Brook Taylor and the Method of Increments," *Archive for History of Exact Sciences* 34 [1] [1984]: 1–140).

59 – The key passage is quoted in the *YuktiBhāṣā* and attributed to the *TantraSangraha* (Thampuran and Aiyar, *YuktiBhāṣā,* pt. 1, p. 190). The passage is *not* to be

found in the *TantraSangraha* of the Trivandrum Sanskrit Series (*Tantrasangraha,* ed. S. K. Pillai, Trivandrum Sanskrit Series, 188 [Trivandrum, 1958]), the English translation of which has recently been serialized in the *Indian Journal of History of Science*. The authors of the modern *YuktiBhāṣā* commentary have, however, used a transcript of the MSS of the *TantraSangrahaVyakhya* in the Desamangalattu Mana, a well-known Namboodri household. This version of the *TantraSangraha* is found in the *TantraSangrahaVyakhya,* Palm Leaf MS No. 697, and its transcript No. T1251, both in the Kerala University MS Library, Trivandrum. The missing verses are after II.21a of the Trivandrum Sanskrit Series MS. The same verses are also found on pp. 68–69 of the transcript No. T-275 of the *TantraSangrahaVyakhya* at Trippunitra Sanskrit College Library, which is copied from the manuscript of the Desamangalattu Mana. See also K. V. Sarma, *A History of the Kerala School of Astronomy (in Perspective)* (Hoshiarpur, 1972), p. 17; A. K. Bag, "Madhava's Sine and Cosine Series," *Indian Journal of History of Science* 11 (1976): 54–57; T. A. Sarasvati Amma, *Geometry in Ancient and Medieval India,* 2d ed. (New Delhi: Motilal Banarsidass, 1999), pp. 184–190. For detailed quotations and a more mathematical account of the passages, see C. K. Raju, "Approximation and Proof in the *YuktiBhāṣā* Derivation of Madhava's Sine Series" (paper presented at the National Seminar on Applied Science in Sanskrit Literature: Various Aspects of Utility, Agra, 20–22 February 1999).

60 – Clavius, cited in note 32 above. The secant tables in Stevin, *The Principal Works of Simon Stevin,* vol. 3, *Astronomy and Navigation,* are similar.

61 – The procedure is described in C. K. Raju, "Kamāl or Rapalagai."

62 – A. P. Jushkevich [Juschkewitsch], *Geschichte der Mathematik in Mittelalter* (Leipzig: Treubner, 1964), German translation of the original (Moscow, 1961).

63 – Victor J. Katz, *A History of Mathematics: An Introduction* (HarperCollins-CollgePublishers, 1992).

64 – C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), p. 114; Carl B. Boyer, *A History of Mathematics* (New York: Wiley, 1989), p. 417.

65 – For example, Edwards, pp. 109–113.

66 – For example, V. I. Arnol'd states that "Newton's basic discovery was that everything had to be expanded in infinite series. . . . Newton, although he did not strictly prove convergence, had no doubts about it. . . . What did Newton do in analysis? What was his main mathematical discovery? Newton invented Taylor series, the main instrument of analysis" (V. I. Arnold, *Huygens and Barrow, Newton and Hooke: Pioneers in Mathematical Analysis and Catastrophe Theory from Evolvents to Quasicrystals,* trans. from the Russian by Eric J. F. Primrose [Basel and Boston: Birkhaüser Verlag, 1990], pp. 35–42).

67 – See the letter by Matteo Ricci cited in note 30 above.

68 – Fermat's challenge problem to European mathematicians, particularly to Wallis, concerned "Pell's" equation. (The name is due to Euler, and Pell had nothing to do with this equation.) Fermat's correspondence with Frenicle explicitly mentions the case $n = 61$ of "Pell's" equation, which has the solution $x = 226153980$, and $y = 176631904$, which is the case that appears in Bhaskara's *Beejgaṇita.* A similar problem had earlier been suggested by Brahmagupta, and Bhaskara II provides the general solution with his *cakravāla* method (D. J. Struik, *A Source Book in Mathematics, 1200–1800* [Cambridge: Harvard University Press, 1969], pp. 29–30; I. S. Bhanu Murthy, *A Modern Introduction to Ancient Indian Mathematics* [New Delhi: Wiley Eastern, 1992]).